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LOW-RANK APPROXIMATION TO HETEROGENEOUS ELLIPTIC PROBLEMS*

GUANGLIAN LI†

Abstract. In this work, we investigate the low-rank approximation of elliptic problems in heterogeneous media by means of Kolmogorov n -width and asymptotic expansion. This class of problems arises in many practical applications involving high-contrast media, and their efficient numerical approximation often relies crucially on certain low-rank structure of the solutions. We provide conditions on the permeability coefficient κ that ensure a favorable low-rank approximation. These conditions are expressed in terms of the distribution of the inclusions in the coefficient κ , e.g., the values, locations, and sizes of the heterogeneous regions. Further, we provide a new asymptotic analysis for high-contrast elliptic problems based on the perfect conductivity problem and layer potential techniques, which allows deriving new estimates on the spectral gap for such high-contrast problems. These results provide theoretical underpinnings for several multiscale model reduction algorithms.

Key words. low-rank approximation, heterogeneous elliptic problems, eigenvalue decays, asymptotic expansion, layer potential technique

AMS subject classifications. 65N30, 65N80, 31A35, 35C15

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1. Introduction. Elliptic problems with heterogeneous coefficients, where the value of the coefficient can vary over several orders of magnitude, arise in many practical applications, e.g., reservoir simulation, subsurface flow, battery modeling, and material sciences [15, 16]. This class of problems is computationally very challenging due to the disparity of scales, which often renders the classical numerical treatment inefficient or even infeasible. In recent years, a number of multiscale model reduction techniques, e.g., multiscale finite element methods, heterogeneous multiscale methods, variational multiscale methods, flux norm approach, generalized multiscale finite element methods, and localized orthogonal decomposition, have been proposed in the literature [24, 13, 25, 5, 14, 30, 28], and they have achieved great success in the efficient and accurate simulation of heterogeneous problems. Conceptually, all these techniques rely crucially on a certain low-rank structure of the solution manifold of the heterogeneous problem, in the sense that the solution can be effectively approximated by a few specialized basis functions. Nonetheless, despite the extensive numerical evidence, the existence of such a low-rank structure has rarely been theoretically established, and the excellent empirical efficiency remains rather mysterious. In this paper, we investigate conditions on the coefficient that ensure a favorable low-rank approximation, thereby providing theoretical underpinnings for related algorithms.

Now we mathematically formulate the problem precisely. Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain with a boundary ∂D . Then we seek a function $u \in V :=$

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$H_0^1(D)$ such that

$$(1.1) \quad \begin{aligned} \mathcal{L}u &:= -\nabla \cdot (\kappa \nabla u) = f && \text{in } D, \\ u &= 0 && \text{on } \partial D, \end{aligned}$$

where the force term $f \in L^2(D)$. The permeability coefficient κ is assumed to be in $L^\infty(D)$ with $\alpha \leq \kappa(x) \leq \beta$ almost everywhere in the domain D for some lower bound $\alpha > 0$ and upper bound $\beta \gg \alpha$. We denote by $\Lambda := \frac{\beta}{\alpha}$ the ratio of these bounds, which reflects the contrast of the coefficient κ . Throughout, let the space $V := H_0^1(D)$ be equipped with the (weighted) inner product $\langle v_1, v_2 \rangle_D = \int_D \kappa \nabla v_1 \cdot \nabla v_2 dx$ and the associated energy norm $\|v\|_{H_\kappa^1(D)}^2 := \langle v, v \rangle_D$, and denote by $W = L^2(D)$, equipped with the usual norm $\|\cdot\|_{L^2(D)}$ and inner product $(\cdot, \cdot)_D$.

The weak formulation for problem (1.1) is to find $u \in V$ such that

$$(1.2) \quad \langle u, v \rangle_D = (f, v)_D \quad \text{for all } v \in V.$$

The Lax–Milgram theorem implies the well-posedness of problem (1.2). We denote by $\mathcal{S} = \mathcal{L}^{-1} : W \rightarrow V$ the solution operator. By the compactness of the Sobolev embedding $V \hookrightarrow W$ [1], the solution operator \mathcal{S} is compact on W . Further, we denote by \mathcal{U} the image of the unit ball in W under the mapping \mathcal{S} , i.e.,

$$(1.3) \quad \mathcal{U} := \{\mathcal{S}(f) : f \in W \text{ with } \|f\|_{L^2(D)} \leq 1\}.$$

Now we can formalize the central property of interest in this work, i.e., the (low-rank) approximation property of the set \mathcal{U} , as follows. Given a tolerance $\delta > 0$, we aim at finding a linear subspace $X_N \subset V$ of dimension N , dependent of δ , satisfying

$$(1.4) \quad \sup_{u \in \mathcal{U}} \inf_{v \in X_N} \|u - v\|_{H_\kappa^1(D)} \leq C\delta,$$

where C denotes a constant independent of N . The (low-rank) approximation in (1.4) underpins the efficiency of numerical techniques for multiscale problems: for a given tolerance δ , the smaller the dimension N of the approximating subspace X_N , the cheaper the effective problem complexity (potentially) becomes. Thus property (1.4) provides a theoretical lower bound on any numerical treatment, and it is of central importance for the theoretical justification of multiscale model reduction algorithms.

Generally, the existence of a low-rank approximation is not a priori ensured. Consider the following example. Let $\kappa = \kappa(\frac{x}{\epsilon})$ for some $0 < \epsilon \ll 1$, i.e., problem (1.1) corresponds to a periodic and rapidly oscillating elliptic operator. It is well known that the eigenvalues of the solution operator \mathcal{S} decay as $\mathcal{O}(n^{-\frac{2}{d}})$ [32, 29]. In particular, this and the discussions in section 3 below (cf. (3.3)) imply that the problem actually does not admit a low-rank approximation for higher dimensions. Thus, a low-rank approximation is not always feasible for every problem.

In this paper, we investigate the situation when a low-rank approximation to problem (1.1) is favorable, especially for high-contrast problems where the contrast $\Lambda \rightarrow \infty$ in some regions [7, 27]. It is well known that when the source term f has high regularity or a special structure, e.g., low-rank expression, there will be a fast decay in the Kolmogorov n -width [31, 12, 22]. In a slightly different context of stochastic homogenization, the recent work [17, Corollary 4] provides a low-rank approximation of a κ -harmonic function that grows at most polynomially at the infinity. This assertion is proved under the assumption that the scalar and vector potentials of the harmonic coordinates in (1.1) grow sublinearly, which holds if the coefficient κ is stationary and

qualitatively ergodic. In this paper, we will not make use of special assumptions on the source term f . The focus of this work is on *structural conditions of the permeability field* κ that ensure a favorable low-rank structure in the sense of (1.4), in terms of spectral gap in the Kolmogorov n -width.

The contributions of this work are threefold. First, we formulate the main goal (1.4) into the eigenvalue decay estimate of the solution operator \mathcal{S} (cf. Proposition 3.2) and provide one sufficient condition that ensures a favorable low-rank approximation to the corresponding elliptic equations (cf. Proposition 4.1). Second, we give a detailed study on the eigenvalue estimate of the operator \mathcal{S} in the context of heterogeneous media (with piecewise constant high-contrast coefficient). This is achieved by a precise characterization of the dominant eigenmodes in Theorem 5.1 and a novel orthogonal decomposition of the space in Theorem 5.4. To the best of our knowledge, there is no known analogous result on the eigenvalue estimate in the literature. Third and last, based on the aforementioned decay estimate, layer potential techniques, and the perfect conductivity problem (i.e., the weak H^1 limit of the solution when the contrast $\Lambda \rightarrow \infty$), we derive an accurate asymptotic expansion for the high-contrast case in Theorem 6.6, which improves several known results [7, 8]. In particular, it provides a rather explicit low-rank approximation; cf. Proposition 6.8.

We conclude this section by discussing related results in the literature. So far there are only a few results on the low-rank approximation of heterogeneous elliptic problems in the literature. In [3, Lemma 2.6], a rank N of order $\log(\frac{1}{\delta})$ was given, which estimates locally in L^2 norm for any arbitrary L^∞ -coefficient and any given prescribed error δ . In [21], a local (generalized) finite element basis (i.e., AL basis) was constructed. With H being the mesh width of the finite element mesh, it consists of $\mathcal{O}((\log \frac{1}{H})^{d+1})$ basis functions per nodal point and preserves the convergence rate of the classical finite element method for Poisson-type problems. Nonetheless, these results [3, 21] remain κ dependent and make no specific assumptions on the permeability coefficient κ which are critical for an efficient low-rank approximation. In contrast, in this work, we shall exploit certain structures on the permeability coefficient κ in order to obtain a favorable low-rank approximation.

The paper is organized as follows. Section 2 is an overview of the main results. In section 3, we provide an approximation to Kolmogorov n -width $d_n(\mathcal{S}(W); W)$ and $d_n(\mathcal{S}(W); V)$ via the eigenvalues of the operator \mathcal{S} . This highlights the central role of eigenvalue decay estimate in the analysis. In section 4 we present one sufficient condition for the low-rank approximations to the solutions of some elliptic equations. In section 5, we identify the characteristic of the dominant eigenmodes of the operator \mathcal{S} and thus derive bounds on the leading eigenvalues. In section 6, we derive a new asymptotic expansion for high-contrast problems with the weak limit as the zeroth order approximant and, as a byproduct, also an estimate on the decay of Kolmogorov n -width. Finally, a conclusion is drawn in section 7.

2. Overview of main results and the general proof strategy. In this section, we survey the main results and give the general idea of their proofs. The precise statements and the detailed proofs are deferred to the following sections.

Our first main result characterizes precisely the low-rank approximation error via the eigenvalues λ_n (ordered nonincreasingly) of the solution map \mathcal{S} ; cf. Proposition 3.2. This result is proved via the concepts of Kolmogorov n -width and approximation numbers from classical approximation theory [34, 33]. It highlights the central role of eigenvalue decay/spectral gap in the study of low-rank approximation and motivates us to investigate problem (1.4) by spectral analysis.

RESULT 2.1 (cf. Proposition 3.2). *Let \mathcal{U} be defined in (1.3). There holds*

$$d_n(\mathcal{S}(W); V) := \inf_{X_n} \sup_{y \in \mathcal{U}} \inf_{x \in X_n} \|x - y\|_{H_\kappa^1(D)} = \sqrt{\lambda_{n+1}},$$

where the infimum is taken over all n -dimensional subspace $X_n \subset V$.

By analyzing the error equation more closely (using an a priori elliptic regularity estimate), we derive one sufficient condition for the existence of a low-rank approximation. While this condition itself is not constructive, it motivates the use of multiple local harmonic functions in the domain D for constructing low-rank approximations.

RESULT 2.2 (cf. Proposition 4.1). *Let κ_0 be the mean of κ and u_0 the corresponding solution, and $f \in H^1(D)$. If there are harmonics $\{\phi_i\}_{i=1}^n$ such that*

$$(2.1) \quad \left| u_0 + \sum_{i=1}^n \phi_i \right|_{H^1(D \setminus D_\delta)} \leq \varepsilon^{\frac{1}{3}}, \quad \|\nabla \phi_i\|_{L^\infty(D)} \leq 1, \quad \text{and} \quad \|\phi_i\|_{L^2(D)} \leq \varepsilon$$

for the tolerance $\varepsilon > 0$, then there holds for some $C(D)$ depending only on D that

$$\left| u - \left(u_0 + \eta \sum_{i=1}^n \phi_i \right) \right|_{H^1(D)} \leq C(D) n^2 \varepsilon^{\frac{1}{3}} \left(\frac{\beta}{\alpha^2} \|f\|_{H^1(D)} + \frac{\beta}{\alpha} \right).$$

Let the coefficient κ be piecewise constant with m inclusions; then the full space V can be orthogonally decomposed into four simpler spaces (cf. Theorem 5.4). This decomposition and the minmax principle yield eigenvalue decay rates. Further, by layer potential techniques, we derive a sharp asymptotic expansion using the perfect conductivity problem as the zeroth order approximation in Theorem 6.6, which is of independent interest. Then under the assumption that the Poincaré constant of the perforated problem is negligible, we show the low-rank structure of problem (1.1) in Proposition 6.8.

RESULT 2.3 (informal version of Proposition 6.8). *For certain high-contrast problems, there holds*

$$d_i(\mathcal{S}(W); V) \begin{cases} \geq C_1(D) & \text{for } i \leq m-1; \\ \leq \Lambda^{-1/2} C_2(D) & \text{for } i = m, \end{cases}$$

with constants $C_1(D)$ and $C_2(D)$ depending on the properties of D and its inclusions.

3. Low-rank approximation and eigenvalues. In this section, we show the estimate (1.4) via the definition of Kolmogorov n -width and discuss its relation with the eigenvalues of the solution map \mathcal{S} (with the help of an approximation number). We shall derive two estimates on Kolmogorov n -width in terms of the eigenvalues of \mathcal{S} .

First, we recall the definitions of Kolmogorov n -width and approximation numbers. The Kolmogorov n -width for the solution operator $\mathcal{S}: W \rightarrow W$ is defined by [34, p. 29]

$$(3.1) \quad d_n(\mathcal{S}(W); W) = \inf_{X_n} \sup_{y \in \mathcal{U}} \inf_{x \in X_n} \|x - y\|_{L^2(D)}$$

with the infimum taken over all n -dimensional subspaces $X_n \subset W$. The n -dimensional subspace X_n that attains $d_n(\mathcal{S}(W); W)$ is called the optimal space. The compactness

of \mathcal{S} on W immediately indicates that $d_n(\mathcal{S}(W); W) \rightarrow 0$ as $n \rightarrow \infty$. Since $\mathcal{S} : W \rightarrow V$ is a bounded linear operator, we can have an analogous definition

$$(3.2) \quad d_n(\mathcal{S}(W); V) = \inf_{X_n} \sup_{y \in \mathcal{U}} \inf_{x \in X_n} \|x - y\|_{H_\kappa^1(D)},$$

where the infimum is taken over all n -dimensional subspaces $X_n \subset V$. However, generally, there is no guarantee that $d_n(\mathcal{S}(W); V) \rightarrow 0$ as $n \rightarrow \infty$.

The Kolmogorov n -width $d_n(\mathcal{S}(W); W)$ can be characterized precisely by the spectrum of the operator \mathcal{S} . Since the operator $\mathcal{S} : W \rightarrow W$ is nonnegative, compact, and self-adjoint, by the standard spectral theory [38], it has at most countably many discrete eigenvalues, with zero being the only accumulation point, and each nonzero eigenvalue has only finite multiplicity. Let $\{(\lambda_j, v_j)\}_{j=1}^\infty$ be the eigenvalues and corresponding $L^2(D)$ normalized eigenfunctions of \mathcal{S} listed according to their algebraic multiplicities and the eigenvalues ordered nonincreasingly. Then, the eigenfunctions $\{v_j\}_{j=1}^\infty$ form an orthonormal basis in $L^2(D)$, and $\{\sqrt{\lambda_j} v_j\}_{j=1}^\infty$ form a complete orthonormal system in V . Then an application of Theorem 2.2 of [34, Chapter IV] yields immediately

$$(3.3) \quad d_n(\mathcal{S}(W); W) = \lambda_{n+1}$$

with the subspace $V_n := \text{span}\{v_1, \dots, v_n\}$ being an optimal space for $n = 1, 2, \dots$.

Next we estimate the Kolmogorov n -width $d_n(\mathcal{S}(W); V)$. To this end, we first recall the definition of the approximation number for a bounded linear operator in W . The $(n+1)$ th approximation number [33, section 2.3.1], denoted by $a_{n+1}(\mathcal{S})$, of an operator $\mathcal{S} \in \mathcal{B}(W, W)$ is defined by

$$(3.4) \quad a_{n+1}(\mathcal{S}) := \inf\{\|\mathcal{S} - L\|_{W \rightarrow W} : L \in \mathfrak{F}(W, W), \text{rank}(L) \leq n\},$$

where the notation $\mathfrak{F}(X, Y)$ means the set of all finite-rank operators from X to Y for any two Banach spaces X and Y , and $\|\cdot\|_{W \rightarrow W}$ denotes the operator norm on the space W . The finite rank operator that attains the infimum is called the optimal operator. The approximation number $a_n(\mathcal{S})$ provides a lower bound of the worst-case convergence rate for any finite-rank approximation to \mathcal{S} (in particular, any numerical treatment). The definition of s -numbers [33, section 2.2] implies that $d_n(\mathcal{S}(W); W)$ and $a_n(\mathcal{S})$ are both s -numbers for the compact operator \mathcal{S} . By the uniqueness of s -numbers of any operator between Hilbert spaces [33, section 2.11.9], we deduce

$$(3.5) \quad a_{n+1}(\mathcal{S}) = d_n(\mathcal{S}(W); W) = \lambda_{n+1}.$$

Remark 3.1. The choice of the finite-rank operator in the definition (3.4) is fairly flexible. In particular, assume that D is a bounded, convex polygon and the coefficient $\kappa \in \mathcal{C}^2$. Let L be a finite-rank operator constructed from the conforming P_1 finite element discretization of \mathcal{S} . Then the standard FEM a priori estimate [23, Chapter 4] and (3.4) imply

$$a_{n+1}(\mathcal{S}) \leq C \Lambda n^{-\frac{2}{d}},$$

where C denotes a positive constant independent of α , β , and n .

Our next endeavor is to estimate the Kolmogorov n -width $d_n(\mathcal{S}(W); V)$ in terms of the eigenvalues λ_n . This is achieved by constructing a finite-rank operator to approximate \mathcal{S} directly, then invoking (3.2) to obtain the desired estimate. The finite-rank operator is constructed below. Given $n \in \mathbb{N}_+$, we define an orthogonal projection operator $\Pi_n : V \rightarrow V_n := \text{span}(\{v_i\}_{i=1}^n)$ by

$$(3.6) \quad \langle v - \Pi_n v, \varphi \rangle_D = 0 \quad \text{for all } \varphi \in V_n.$$

Let $\mathfrak{F}(W, V) \ni \mathcal{S}_n := \Pi_n \mathcal{S}$ be a rank $\leq n$ operator. A simple calculation yields

$$\|\mathcal{S} - \mathcal{S}_n\|_{W \rightarrow W} = \lambda_{n+1}.$$

First we state an a priori estimate on the projection operator Π_n .

LEMMA 3.1. *Let u be the solution to (1.1). For the projection operator Π_n defined in (3.6), there holds*

$$(3.7) \quad \|u - \Pi_n u\|_{H_\kappa^1(D)} \leq \sqrt{\lambda_{n+1}} \|f\|_{L^2(D)}.$$

Proof. Since $\{v_j\}_{j=1}^\infty$ and $\{\sqrt{\lambda_j} v_j\}_{j=1}^\infty$ form orthonormal bases in $L^2(D)$ and V , respectively, for any $u \in V \subset L^2(D)$, there exists a sequence $\{c_j\}_{j=1}^\infty \in \ell^2$ such that $u = \sum_{j=1}^\infty c_j v_j$ and $\Pi_n u = \sum_{j=1}^n c_j v_j$ by the definition (3.6), which gives directly $c_j = \lambda_j \langle u, v_j \rangle_D$. Further, we have

$$(3.8) \quad \begin{aligned} \|u - \Pi_n u\|_{L^2(D)}^2 &= \sum_{j=n+1}^\infty c_j^2 = \sum_{j=n+1}^\infty \frac{\lambda_j}{\lambda_j} c_j^2 \\ &\leq \lambda_{n+1} \sum_{j=n+1}^\infty \frac{1}{\lambda_j} c_j^2 = \lambda_{n+1} \|u - \Pi_n u\|_{H_\kappa^1(D)}^2. \end{aligned}$$

By taking $v = (u - \Pi_n u)$ as the test function in (1.2) and applying (3.6), we obtain

$$\|u - \Pi_n u\|_{H_\kappa^1(D)}^2 = (f, u - \Pi_n u)_D.$$

Now the desired assertion follows from (3.8) and the Cauchy–Schwarz inequality. \square

Remark 3.2. The condition $f \in L^2(D)$ is essential for obtaining the convergence rate in Lemma 3.1. If $f \in H^{-1}(D)$ only, the estimate (3.7) is generally not true.

Now we can derive the main result of this section.

PROPOSITION 3.2. *The rank $\leq n$ operator $\mathcal{S}_n := \Pi_n \mathcal{S}$ is an optimal operator to the solution operator \mathcal{S} for $n \in \mathbb{N}_+$. There holds*

$$d_n(\mathcal{S}(W); V) = \sqrt{\lambda_{n+1}}.$$

Proof. The stated identity is equivalent to

$$\sqrt{\lambda_{n+1}} \leq d_n(\mathcal{S}(W); V) \leq \sqrt{\lambda_{n+1}}.$$

The upper bound follows directly from Lemma 3.1, the definition (3.2), and the orthogonality (3.6). Next, we show its lower bound via the definition (3.2). Given any n -dimensional linear subspace $X_n \subset V$, since $\dim(V_{n+1}) = n+1 > n = \dim(X_n)$ and $V_{n+1} \subset V$, [26, Lemma 2.3] implies the existence of a vector $v \in V_{n+1}$, satisfying

$$(3.9) \quad \text{dist}(v, X_n) := \inf_{w \in X_n} \|v - w\|_{H_\kappa^1(D)} = \|v\|_{H_\kappa^1(D)} > 0.$$

Since $\{\sqrt{\lambda_j} v_j\}_{j=1}^{n+1}$ form an orthonormal basis in V_{n+1} , the element $v \in V_{n+1}$ admits the expansion $v := \sum_{j=1}^{n+1} \lambda_j \langle v, v_j \rangle_D v_j$ and

$$(3.10) \quad \|v\|_{H_\kappa^1(D)}^2 = \sum_{j=1}^{n+1} \lambda_j |\langle v, v_j \rangle_D|^2 \geq \lambda_{n+1} A^2$$

with $A := (\sum_{j=1}^{n+1} |\langle v, v_j \rangle_D|^2)^{1/2}$. The last inequality is due to the nonincreasing property of the eigenvalues.

Let $f := A^{-1} \sum_{j=1}^{n+1} \langle v, v_j \rangle_D v_j$; then we obtain $\|f\|_{L^2(D)} = 1$ and $\mathcal{S}f = A^{-1}v$. A direct calculation leads to

$$\|\mathcal{S}f\|_{H_\kappa^1(D)}^2 := \|A^{-1}v\|_{H_\kappa^1(D)}^2 = A^{-2} \|v\|_{H_\kappa^1(D)}^2 \geq \lambda_{n+1},$$

where the last inequality follows from (3.10). In view of (3.9), we derive

$$\inf_{w \in X_n} \|w - A^{-1}v\|_{H_\kappa^1(D)} := \text{dist}(A^{-1}v, X_n) = \|\mathcal{S}f\|_{H_\kappa^1(D)} \geq \sqrt{\lambda_{n+1}},$$

and this gives the desired lower bound. \square

Lemma 3.1 (and Proposition 3.2) implies that V_n is the optimal space for approximating solutions to problem (1.1) and the convergence rate in V_n is essentially determined by either the eigenvalue decay rate of the solution operator \mathcal{S} or the existence of a spectral gap. Here a spectral gap means that there is an integer $L \in \mathbb{N}_+$ and $0 < \varepsilon \ll 1$ such that

$$(3.11) \quad \begin{aligned} d_1(\mathcal{S}(W); V) &\geq d_2(\mathcal{S}(W); V) \geq \cdots \geq d_L(\mathcal{S}(W); V) \\ &\gg \varepsilon \geq d_{L+1}(\mathcal{S}(W); V) \geq \cdots \end{aligned}$$

The identity (3.3) and Proposition 3.2 both highlight the central role of the eigenvalue decay/spectral gap in the study of the low-rank approximation of heterogeneous elliptic problems: a fast eigenvalue decay or spectral gap implies that the solution operator can be well approximated by a small set of basis functions. We shall analyze the spectral gap for elliptic problems in high-contrast media in sections 5 and 6. Before that, we first provide one sufficient condition that ensures the low-rank structure.

4. One sufficient condition for low-rank approximation. In this part, we provide one sufficient condition for the low-rank approximation to problem (1.1) via its error equation, for the case of a bounded contrast Λ .

To motivate the construction, we begin with a simple situation. Given a prescribed tolerance $\varepsilon > 0$, let κ_0 be an approximation to the permeability coefficient κ (e.g., on a coarse mesh) and u_0 be the solution to problem (1.1) with κ_0 in place of κ (assuming also $\alpha \leq \kappa_0 \leq \beta$). Then the following implication holds:

$$(4.1) \quad \text{If } \|\kappa - \kappa_0\|_{L^\infty(D)} \leq \varepsilon, \text{ then } |u - u_0|_{H^1(D)} \leq \varepsilon \alpha^{-2} C_{\text{poin}}(D) \|f\|_{L^2(D)}$$

with $C_{\text{poin}}(D)$ being the Poincaré constant for the domain D and $|\cdot|_{H^1(\omega)}$ denoting the $H^1(\omega)$ -seminorm on $\omega \subset D$. This assertion can be verified directly by a perturbation argument and the a priori estimate for elliptic problems with rough coefficient as follows. The equation for the difference $u - u_0 \in V$ is given by

$$-\nabla \cdot (\kappa \nabla (u - u_0)) = \nabla \cdot ((\kappa - \kappa_0) \nabla u_0) \quad \text{in } D.$$

This equation together with the coercivity of the elliptic problem yields

$$\begin{aligned} \alpha |u - u_0|_{H^1(D)}^2 &\leq \langle u - u_0, u - u_0 \rangle_D = - \int_D (\kappa - \kappa_0) \nabla u_0 \cdot \nabla (u - u_0) dx \\ &\leq \|\kappa - \kappa_0\|_{L^\infty(D)} |u_0|_{H^1(D)} |u - u_0|_{H^1(D)} \\ &\leq C_{\text{poin}}(D) \varepsilon \alpha^{-1} \|f\|_{L^2(D)} |u - u_0|_{H^1(D)}, \end{aligned}$$

and the assertion (4.1) follows by dividing $\alpha |u - u_0|_{H^1(D)}$ from both sides. In the last line we have employed Hölder's inequality and the following a priori estimate:

$$\alpha |u_0|_{H^1(D)}^2 \leq \|f\|_{L^2(D)} \|u_0\|_{L^2(D)} \leq \|f\|_{L^2(D)} C_{\text{poin}}(D) |u_0|_{H^1(D)}.$$

Our focus in the rest of this section is to relax the condition in (4.1). Then in addition to the term u_0 , extra basis functions are needed in order to get a good approximation. We shall analyze one specific situation that generalizes assertion (4.1). Let

$$(4.2) \quad \kappa_0 = \oint_D \kappa(x) dx := \frac{1}{|D|} \int_D \kappa(x) dx$$

be a zeroth-order approximation to the permeability field κ . Accordingly, we define $u_0 \in V$ to be the corresponding solution to the problem

$$(4.3) \quad -\nabla \cdot (\kappa_0 \nabla u_0) = f \quad \text{in } D.$$

For any given $\delta > 0$, let $D_\delta = \{x \in D : \text{dist}(x, \partial D) \leq \delta\}$. Further, let χ be a cutoff function on the domain D satisfying $\chi = 1$ in $D \setminus D_\delta$, $\chi = 0$ on ∂D , $0 \leq \chi \leq 1$, and $\|\nabla \chi\|_{L^\infty(D)} \leq \delta^{-1}$. Now we can give a sufficient condition for the existence of a low-rank approximation. The construction is based on certain harmonic functions in the interior of the domain D .

PROPOSITION 4.1. *Let $d \leq 3$, $f \in H^1(D)$, $\varepsilon > 0$ be a given tolerance, and let κ_0 and u_0 be defined in (4.2) and (4.3), respectively. Further, assume that there are harmonic functions $\{\phi_i\}_{i=1}^n$ for some $n \in \mathbb{N}_+$ satisfying (2.1). Then there holds for some constant depending only on the domain D*

$$\left| u - \left(u_0 + \chi \sum_{i=1}^n \phi_i \right) \right|_{H^1(D)} \leq C(D) n^2 \varepsilon^{\frac{1}{3}} \left(\frac{\beta}{\alpha^2} \|f\|_{H^1(D)} + \frac{\beta}{\alpha} \right).$$

Proof. Let $v = u - (u_0 + \chi \sum_{i=1}^n \phi_i)$. Clearly $v = 0$ on ∂D . Using the governing equations (1.1) and (4.3), and noting that the functions ϕ_i s are harmonic, we deduce that the difference v satisfies

$$\begin{aligned} \tilde{f} &:= -\nabla \cdot (\kappa \nabla v) = f + \nabla \cdot (\kappa \nabla u_0) + \sum_{i=1}^n \nabla \cdot (\kappa \nabla (\chi \phi_i)) \\ &= f + \nabla \cdot ((\kappa - \kappa_0 + \kappa_0) \nabla u_0) + \sum_{i=1}^n \nabla \cdot ((\kappa - \kappa_0) \nabla (\chi \phi_i)) - \sum_{i=1}^n \nabla \cdot (\kappa_0 \nabla ((1 - \chi) \phi_i)) \\ &= \nabla \cdot \left((\kappa - \kappa_0) \nabla \left(u_0 + \chi \sum_{i=1}^n \phi_i \right) \right) - \sum_{i=1}^n \nabla \cdot (\kappa_0 \nabla ((1 - \chi) \phi_i)). \end{aligned}$$

Next we estimate the residual \tilde{f} . By Hölder's inequality, we obtain

$$\begin{aligned} \left| \int_D \tilde{f} v dx \right| &\leq \int_D \left| (\kappa - \kappa_0) \nabla \left(u_0 + \chi \sum_{i=1}^n \phi_i \right) \cdot \nabla v \right| dx + \sum_{i=1}^n \int_D |\kappa_0 \nabla ((1 - \chi) \phi_i) \cdot \nabla v| dx \\ &\leq \beta \left(\left| u_0 + \chi \sum_{i=1}^n \phi_i \right|_D + \sum_{i=1}^n |(1 - \chi) \phi_i|_D \right) \cdot |v|_{H^1(D)}. \end{aligned}$$

It remains to bound the two terms in the bracket. For the first term, we appeal to the splitting

$$\left| u_0 + \chi \sum_{i=1}^n \phi_i \right|_D^2 = \left| u_0 + \sum_{i=1}^n \phi_i \right|_{D \setminus D_\delta}^2 + \left| u_0 + \chi \sum_{i=1}^n \phi_i \right|_{D_\delta}^2 := \text{I} + \text{II},$$

where the first term I is bounded by $\varepsilon^{\frac{2}{3}}$, by Assumption (2.1). To bound the second term II, we apply Young's inequality

$$\text{II} \leq 3 \int_{D_\delta} |\nabla u_0|^2 dx + 3 \int_{D_\delta} \left| \chi \sum_{i=1}^n \nabla \phi_i \right|^2 dx + 3 \int_{D_\delta} \left| \nabla \chi \sum_{i=1}^n \phi_i \right|^2 dx = 3 \sum_{j=1}^3 \text{II}_j.$$

To bound the term II_1 , we employ a corollary of the following a priori estimate on u_0 [23, Theorem 3.1.2.1] and Sobolev embedding $H^3(D) \hookrightarrow W^{1,\infty}(D)$ when $d \leq 3$ that

$$\|\nabla u_0\|_{L^\infty(D)} \leq C(D) \alpha^{-1} \|f\|_{H^1(D)}$$

for some constant $C(D)$ depending on the domain D . Throughout this proof, $C(D)$ denotes a constant depending only on D . Upon noting $|D_\delta| \leq C(D)|\delta|$, we have

$$\text{II}_1 \leq \|\nabla u_0\|_{L^\infty(D)}^2 |D_\delta| \leq C(D) \alpha^{-2} \|f\|_{H^1(D)}^2 \delta.$$

Next by the property of the cutoff function χ and the bounds $\|\nabla \phi_i\|_{L^\infty(D)} \leq 1$ (cf. Assumption (2.1)), we have

$$\text{II}_2 \leq \left(\|\chi\|_{L^\infty(D)} \sum_{i=1}^n \|\nabla \phi_i\|_{L^\infty(D)} \right)^2 |D_\delta| \leq n^2 C(D) \delta.$$

For the third term II_3 , we appeal to the property of the cutoff function again,

$$\text{II}_3 \leq \left(\|\nabla \chi\|_{L^\infty(D)} \sum_{i=1}^n \|\phi_i\|_{L^2(D)} \right)^2 \leq n^2 \varepsilon^2 \delta^{-2}.$$

Combining the preceding three estimates yields

$$\text{II} \leq C(D) n^2 \left(\delta \left(\frac{1}{\alpha^2} \|f\|_{H^1(D)}^2 + 1 \right) + \frac{\varepsilon^2}{\delta^2} \right).$$

Similarly, from Assumption 2.1, we derive

$$\begin{aligned} |(1-\chi)\phi_i|_{H^1(D)}^2 &= |(1-\chi)\phi_i|_{H^1(D_\delta)}^2 \leq 2 \left(\int_{D_\delta} |(1-\chi)\nabla \phi_i|^2 dx + \int_{D_\delta} |(\nabla \chi)\phi_i|^2 dx \right) \\ &\leq 2 \left(C(D) \delta + \frac{\varepsilon^2}{\delta^2} \right). \end{aligned}$$

Taking $\delta = \varepsilon^{\frac{2}{3}}$ yields

$$\int_D \kappa |\nabla v|^2 dx = \left| \int_D \tilde{f} v dx \right| \leq C(D) n^2 \varepsilon^{\frac{1}{3}} \left(\beta \alpha^{-1} \|f\|_{H^1(D)} + \beta \right) |v|_{H^1(D)},$$

which implies directly the desired result, since κ is bounded from below by α . \square

Remark 4.1. The condition (2.1) implicitly imposes a certain regularity on the domain D . The condition $\partial D \in C^{3,a}$, $0 < a < 1$, is sufficient. The requisite number n of harmonic basis functions is problem dependent. For problems with a periodic structure, by the homogenization theory, n can be taken to be $n = d$ [39].

Proposition 4.1 gives one sufficient condition (2.1) for problem (1.1) to admit a low-rank approximation. Under condition (2.1), the triangle inequality gives

$$|u|_{H^1(D \setminus D_\delta)} \leq C(D) n^2 \varepsilon^{\frac{1}{3}} \beta \alpha^{-1} \left(\alpha^{-1} \|f\|_{H^1(D)} + 1 \right).$$

The condition (2.1) actually imposes certain (implicit) structural assumptions on the permeability field κ . Though Proposition 4.1 gives one sufficient condition, it is unfortunately not constructive in nature, and the precise assumption on the permeability field κ is not transparent. Nonetheless, it motivates further analysis by constructing specialized harmonic functions within the domain. In the rest of this paper, we focus on the elliptic operator with high-contrast piecewise constant coefficients κ , for which the dominant eigenmodes can be identified and eigenvalue estimates in the spirit of Proposition 4.1 can be derived. Specifically, we make the following structural assumptions on the domain D and the coefficient κ .

ASSUMPTION 4.1 (structure of D and κ_η). *Let D be a domain with a $C^{2,a}$ ($0 < a < 1$) boundary ∂D , and let $\{D_i\}_{i=1}^m \subset D$ be m pairwise disjoint strictly convex open subsets, each with a $C^{2,a}$ boundary $\Gamma_i := \partial D_i$, and denote $D_0 = D \setminus \bigcup_{i=1}^m \overline{D_i}$. Further, there exists an open set $\omega \subset D$, such that $\bigcup_{i=1}^m D_i \subset \omega$ and $\text{dist}(\partial\omega, \partial D) \geq \tau$, for some $\tau > 0$. Let the permeability coefficient κ_η be piecewise constant defined by*

$$(4.4) \quad \kappa_\eta = \begin{cases} \eta_i & \text{in } D_i, \\ 1 & \text{in } D_0. \end{cases}$$

Let $\eta_{\min} := \min_i \{\eta_i\} \geq 1$.

Throughout, we always take 1 and ϵ_i as the diameters of D and D_i , respectively. Let $\eta = (\eta_1, \dots, \eta_m)$ and $\epsilon = (\epsilon_1, \dots, \epsilon_m)$. Denote $\tau_i := \text{dist}(D_i, \partial D)$ and $\delta_j := \min_{i \neq j} \{\text{dist}(D_i, D_j)\}$. We assume that $\tau_j \geq \delta_j$ for $j = 1, 2, \dots, m$. Without loss of generality, we may relabel the indices for the inclusions D_j such that $|D_1| \geq |D_2| \geq \dots \geq |D_m|$. Further, we use the notation $A \lesssim B$ if $A \leq CB$ for some constant C independent of ϵ_i , η_i , δ_i , and τ_i . The notation $C_{\text{poin}}(\omega)$ denotes the Poincaré constant in the subdomain $\omega \subset D$ for all functions in $H_0^1(\omega)$, i.e.,

$$C_{\text{poin}}(\omega)^2 = \sup_{v \in H_0^1(\omega)} \int_\omega v^2 dx / \int_\omega |\nabla v|^2 dx.$$

A scaling argument shows that $C_{\text{poin}}(\omega) \lesssim \text{diam}(\omega)$.

Below, we denote by $n_i(x)$ the unit outward normal (relatively to D_i) to the interface Γ_i at the point $x \in \Gamma_i$. For a function w defined on $\mathbb{R}^2 \setminus \Gamma_i$ for $i = 1, 2, \dots, m$, we define for $x \in \Gamma_i$,

$$w(x)|_\pm := \lim_{t \rightarrow 0^+} w(x \pm tn_i(x)) \quad \text{and} \quad \frac{\partial}{\partial n_i^\pm} w(x) := \lim_{t \rightarrow 0^+} (\nabla w(x \pm tn_i(x)) \cdot n_i(x))$$

if the limit on the right-hand side exists. We denote by $[w]$ the jump of w across the interface Γ_i defined by

$$[w(x)] := \lim_{t \rightarrow 0^+} (w(x + tn_i(x)) - w(x - tn_i(x))) \quad \text{and} \quad \left[\kappa_\eta \frac{\partial w}{\partial n_i} \right] := \frac{\partial w}{\partial n_i^+} - \eta_i \frac{\partial w}{\partial n_i^-}.$$

5. Eigenvalue decay rate. In this section, we establish the eigenvalue estimates for the operator \mathcal{S} through the maxmin principle and a novel orthogonal decomposition of the space V . Specifically, we seek $\{(v_n, \lambda_n)\} \in V \times \mathbb{R}$ such that

$$(5.1) \quad \begin{cases} \mathcal{S}v_n = \lambda_n v_n & \text{in } D, \\ v_n = 0 & \text{on } \partial D. \end{cases}$$

The weak formulation for the eigenvalue problem is to find $(v_n, \lambda_n) \in V \times \mathbb{R}$ such that

$$(v_n, \phi)_D = \lambda_n \langle v_n, \phi \rangle_D \quad \text{for all } \phi \in V.$$

One approach to characterize the eigenvalues $\{\lambda_n\}_{n=1}^\infty$ is the Rayleigh quotient

$$(5.2) \quad R(v) = \frac{(v, v)_D}{\langle v, v \rangle_D} := \frac{\int_D v^2 dx}{\int_D \kappa |\nabla v|^2 dx}.$$

As a corollary of the maxmin principle, there holds

$$(5.3) \quad \lambda_n = \max_{\substack{V_n \subset V \\ \dim(V_n) \leq n}} \min_{v \in V_n} R(v) = R(v_n).$$

First, we show that piecewise harmonic functions v with high oscillations on the interface Γ_i for $i = 1, 2, \dots, m$ generate unimportant eigenmodes, i.e., the value of the Rayleigh quotient $R(v)$ is small. For simplicity, let $d = 2$ and $D_i := B(O_i, \epsilon_i)$ be balls centering at O_i with radius ϵ_i . Then the set of functions

$$\{\cos k\theta, \sin(k+1)\theta, \ k = 0, 1, \dots\}$$

forms an orthogonal basis of $H^{\frac{1}{2}}(\Gamma_i)$, where the angle θ is with respect to O_i .

THEOREM 5.1. *Let $d = 2$, $D_i := B(O_i, \epsilon_i)$, and $v \in V$ satisfy*

$$-\Delta v = 0 \quad \text{in } D \setminus \bigcup_{i=1}^m \Gamma_i.$$

If $v = \sin k_i \theta$ on the interface Γ_i , where $k_i \in \mathbb{N}_+$ and $i = 1, \dots, m$, then there holds

$$R(v) \leq \frac{1}{\pi \eta_{\min} \sum_{i=1}^m k_i}.$$

Proof. It can be verified directly that $v(x) = (\frac{|x-O_i|}{\epsilon_i})^{k_i} \sin k_i \theta$ in D_i for $i = 1, 2, \dots, m$. Hence, a direct calculation together with Dirichlet's principle [11] and the maximum principle yields

$$\text{for all } i = 1, \dots, m : \pi k_i = |v|_{H^1(D_i)}^2 \quad \text{and} \quad (v, v)_D \leq |D| \leq 1.$$

Thus we obtain

$$R(v) = \frac{(v, v)_D}{\langle v, v \rangle_D} \leq \frac{1}{\sum_{i=1}^m \pi k_i \eta_i} \leq \frac{1}{\pi \eta_{\min} \sum_{i=1}^m k_i},$$

and the desired estimate follows. \square

Theorem 5.1 indicates that, in the high-contrast limit $\eta \rightarrow \infty$, the dominant piecewise harmonic eigenfunctions in (5.1) must have low oscillations on the interfaces $\{\Gamma_i\}_{i=1}^m$. This observation suggests itself a constructive approach to retrieve the dominant eigenfunctions of \mathcal{S} . Specifically, we define auxiliary functions on the domain D that are piecewise constant on $\cup_{j=1}^m D_j$: $\{w_i\}_{i=1}^m \subset H_0^1(D)$ satisfying

$$(5.4) \quad \begin{cases} -\Delta w_i = 0 & \text{in } D \setminus \cup_i \Gamma_i, \\ w_i = \delta_{ik} & \text{on } \Gamma_k, \quad k = 1, 2, \dots, m, \\ w_i = 0 & \text{on } \partial D, \end{cases}$$

where δ_{ik} is the Kronecker delta. The well-posedness of problem (5.4) can be established by a variational method [2]. Below, we provide some a priori estimates on w_i , which are useful for deriving the lower bound of the Rayleigh quotient $R(w_i)$.

LEMMA 5.2. *For $i = 1, 2, \dots, m$, there holds*

$$(5.5) \quad \int_{D_0} |\nabla w_i|^2 dx \leq \begin{cases} \pi(1 + 4\frac{\epsilon_i}{\delta_i}) & \text{if } d = 2, \\ \frac{4}{3}\pi(\frac{1}{2}\delta_i + 3\epsilon_i + 6\frac{\epsilon_i^2}{\delta_i}) & \text{if } d = 3. \end{cases}$$

Proof. We denote by O_i the center of D_i and $B(O_i, \frac{1}{2}\delta_i + \epsilon_i)$ a ball centering at O_i with radius $(\frac{1}{2}\delta_i + \epsilon_i)$. Then $D_i \subset B(O_i, \frac{1}{2}\delta_i + \epsilon_i)$ and $D_j \cap B(O_i, \frac{1}{2}\delta_i + \epsilon_i) = \emptyset$ for $j \neq i$. Further, we define a cutoff function $\rho_i \in \mathcal{C}^2(D)$ by

$$\rho_i(x) = \begin{cases} 1, & x \in B(O_i, \epsilon_i), \\ 0, & x \in D \setminus B(O_i, \frac{1}{2}\delta_i + \epsilon_i), \\ \text{affine} & \text{otherwise.} \end{cases}$$

By construction, $0 \leq \rho_i \leq 1$, $\|\nabla \rho_i\|_{L^\infty(D)} \leq 2/\delta_i$, and $\rho_i = w_i$ on ∂D_0 . The Dirichlet's principle [11] implies

$$\int_{D_0} |\nabla w_i|^2 dx \leq \int_{D_0} |\nabla \rho_i|^2 dx.$$

Together with the identity

$$|B(O_i, \frac{1}{2}\delta_i + \epsilon_i) \setminus B(O_i, \epsilon_i)| = \begin{cases} \pi(\frac{1}{4}\delta_i^2 + \epsilon_i\delta_i) & \text{if } d = 2, \\ \frac{4}{3}\pi(\frac{1}{8}\delta_i^3 + \frac{3}{4}\epsilon_i\delta_i^2 + \frac{3}{2}\epsilon_i^2\delta_i) & \text{if } d = 3, \end{cases}$$

we immediately obtain

$$\int_{D_0} |\nabla w_i|^2 dx \leq \int_{D_0} |\nabla \rho_i|^2 dx \leq \|\nabla \rho_i\|_{L^\infty(D)}^2 |B(O_i, (\delta_i + \epsilon_i)) \setminus B(O_i, \epsilon_i)|.$$

Combining the preceding two estimates shows the desired result. \square

Now we can derive a lower bound on the Rayleigh quotient $R(w_i)$ for $i = 1, 2, \dots, m$.

THEOREM 5.3. *For $i = 1, 2, \dots, m$, there holds*

$$(5.6) \quad R(w_i) \geq \begin{cases} [\pi(1 + 4\frac{\epsilon_i}{\delta_i})]^{-1} |D_i| & \text{if } d = 2, \\ [\frac{4}{3}\pi(\frac{1}{2}\delta_i + 3\epsilon_i + 6\frac{\epsilon_i^2}{\delta_i})]^{-1} |D_i| & \text{if } d = 3. \end{cases}$$

Proof. By definition of $R(w_i)$ and the fact that $w_i \equiv 1$ in D_i , we have

$$R(w_i) := \frac{\int_D w_i^2 dx}{\int_D \kappa |\nabla w_i|^2 dx} \geq \frac{|D_i|}{\int_{D_0} |\nabla w_i|^2 dx}.$$

Then Lemma 5.2 implies the assertion. \square

Remark 5.1. The spatial dimension d impacts the lower bound on $R(w_i)$: in three dimensions, the factor δ_i^{-1} enters the estimate, whereas in two dimensions, it is a constant factor 1 if $\epsilon_i \ll \delta_i$.

To estimate the eigenvalues $\{\lambda_n\}_{n=1}^\infty$ by the maxmin principle, we also need an upper bound on the Rayleigh quotient $R(v)$. To this end, we appeal to a novel orthogonal decomposition of the full space $(V; \langle \cdot, \cdot \rangle_D)$. It is motivated by the dominant modes of the perfect conductivity problem (6.1) in section 6 below, which represents the limit problem when $\eta \rightarrow \infty$.

THEOREM 5.4. *There holds the orthogonal decomposition of the space $(V; \langle \cdot, \cdot \rangle_D)$:*

$$(5.7) \quad V := V_m \oplus V^b \oplus V_0^b \oplus V^h.$$

The subspaces V_m , V^b , V_0^b , and V^h are defined by $V_m = \text{span}(\{w_i\}_{i=1}^m)$, $V^b = \{v \in V : v = 0 \text{ in } \bar{D}_0\}$, $V_0^b = \{v \in V : v = 0 \text{ in } \cup_{i=1}^m \bar{D}_i\}$, and $V^h = \{v \in V : -\Delta v = 0 \text{ in } D \setminus \cup_{j=1}^m \Gamma_j \text{ and } \int_{\Gamma_i} \frac{\partial v}{\partial n_i^+} ds(x) = 0 \text{ for } i = 1, 2, \dots, m\}$, respectively.

Proof. The orthogonality of the spaces V_m , V^b , and V_0^b can be shown directly. Indeed, first, the orthogonality of V^b and V_0^b is trivial since their supports are disjoint. Second, since the functions in V^b are supported in $\cup_{i=1}^m D_i$, where V_m is piecewise constant, V^b is orthogonal to V_m in $(V; \langle \cdot, \cdot \rangle_D)$. Third, with $v \in V_0^b$, the divergence theorem implies

$$\langle v, w_i \rangle_D = \int_{D_0} \nabla v \cdot \nabla w_i dx = - \sum_{j=1}^m \int_{\Gamma_j} \frac{\partial w_i}{\partial n_j^+} v ds(x) = 0.$$

Let $\tilde{V} := V_m \oplus V^b \oplus V_0^b$. Then these discussions indicate that (5.7) is equivalent to

$$(5.8) \quad V^h = \tilde{V}^\perp := \{v \in V : \langle v, w \rangle_D = 0 \text{ for all } w \in \tilde{V}\}.$$

To complete the proof, we only need to show (5.8). The proof consists of two steps.

Step 1. We show that the inclusion $V^h \subset \tilde{V}^\perp$. For any $v \in V^h$, by definition, $v \in H_A(D_j)$ for $j = 0, 1, \dots, m$, where $H_A(D_j) := \{v \in V : -\Delta v = 0 \text{ in } D_j\}$. Thus, $v \in V^{b\perp}$ and $v \in V_0^{b\perp}$. It suffices to prove $\langle v, w \rangle_D = 0$ for all $w \in V_m$. Actually, since w is constant in each inclusion D_i for $i = 1, 2, \dots, m$ and $v \in H_A(D_0)$, the divergence theorem leads directly to

$$\langle v, w \rangle_D = \int_{D_0} \nabla v \cdot \nabla w dx = \sum_{i=1}^m w|_{\Gamma_i} \int_{\Gamma_i} \frac{\partial}{\partial n_i^+} v ds(x) = 0,$$

where the last identity follows from the definition of the space V^h .

Step 2. We show that the inclusion $V^h \supset \tilde{V}^\perp$. For any $v \in \tilde{V}^\perp$, we have $v \in V^{b\perp}$ and $v \in V_0^{b\perp}$. This indicates $v \in H_A(D_j)$ for $j = 0, 1, \dots, m$. Then $v \in V_m^\perp$ yields $\int_{\Gamma_j} \frac{\partial v}{\partial n_j^+} ds(x) = 0$ and this completes the proof. \square

By Theorem 5.3, the functions in the m -dimensional subspace V_m constitute the dominant eigenmodes. Further, in section 6 (cf. Remark 6.1), we will show

$$(5.9) \quad R(v) \lesssim \eta_{\min}^{-1} \quad \text{for all } v \in V^h \text{ when } \eta \rightarrow \infty.$$

Thus it suffices to estimate the Rayleigh quotient $R(v)$ for $v \in V^b \oplus V_0^b$ to obtain the eigenvalue estimate, which will be discussed next separately.

For $v \in V^b$, an application of the Poincaré inequality in each inclusion D_i yields

$$\int_{D_i} |v|^2 dx \leq C_{\text{poin}}(D_i)^2 \int_{D_i} |\nabla v|^2 dx \quad \text{for } v \in V^b.$$

This together with the characterization of the space V^b implies

$$R(v) \leq \max_i \{\eta_i^{-1} C_{\text{poin}}(D_i)^2\} \quad \text{for } v \in V^b.$$

That is, in the high-contrast limit, the contribution of the space V^b to the Rayleigh quotient $R(v)$ is negligible and will not contribute much to the dominant eigenmodes.

It remains to estimate the contribution of V_0^b to the Rayleigh quotient $R(v)$. Note that V_0^b represents the solution space of the degenerate elliptic problem with holes in the domain and a homogeneous Dirichlet boundary condition [36]. To the best of our knowledge, in this case, the Rayleigh quotient $R(v)$ exhibits fairly complex behavior and is still not fully understood, except in the following two scenarios. The first result [9] we are aware of is in the case that every compact set $K \subset D$ belongs to D_0 if the size of the inclusion ϵ is small enough, for which, there holds $\max_{v \in V_0^b} R(v) \leq C_{\text{poin}}(D)^2$. This indicates that there exist many important modes in the space V_0^b , since the eigenvalues of the inverse Laplacian in D decay as $\mathcal{O}(n^{-\frac{2}{d}})$, and thus the problem does not admit a low-rank structure. The second result asserts that $R(v) \rightarrow 0$ for all $v \in V_0^b$ if the characteristic function of the set of inclusions $\cup_{i=1}^m D_i$ weakly \star converges to a strictly positive function in $L^\infty(D)$ as $\epsilon \rightarrow 0$ [36, Chapter 15]. Thus, the functions in V_0^b contribute negligibly to the Rayleigh quotient $R(v)$.

In this paper, we are mainly interested in the spectral gap, which implies a low-rank structure in V_0^b . Thus, we make the following assumption on the Poincaré constant $C_{\text{poin}}(D_0)$ of the perforated domain D_0 .

ASSUMPTION 5.1 (Poincaré constant in the perforated domain D_0).

$$C_{\text{poin}}(D_0)^2 \ll \min_{i=1, \dots, m} \{R(w_i)\}.$$

Now we can state an upper bound on the $(m+1)$ th eigenvalue λ_{m+1} .

THEOREM 5.5. *The following statements hold:*

- (a) *Let $\epsilon \rightarrow 0$, $\eta \rightarrow \infty$ and that $\cup_{i=1}^m D_i$ are periodically embedded into the global domain D . Then there holds*

$$\lambda_{m+1} \lesssim \min_i \{\epsilon_i^2\}.$$

- (b) *Fix ϵ . Let $\epsilon_i \leq \frac{1}{2}\delta_i$ and $\eta \rightarrow \infty$, and let Assumption 5.1 hold. Then there holds*

$$\lambda_{m+1} \ll \lambda_m.$$

Proof. In either case, the dominant modes lie in the spaces $V_m \oplus V_0^b$. In the periodic setting (a), due to [35, Appendix, Lemma 1], there holds

$$R(v) \leq C(D_0)\epsilon_i^2.$$

This and the maxmin principle (5.3) yield the desired assertion. Case (b) follows directly from Assumption 5.1. \square

Theorem 5.5 provides a highly desirable spectral gap, under the designate conditions on the inclusions, i.e., the coefficient is periodic with $\epsilon_i \rightarrow 0$ or the perforated domain D_0 satisfies a suitable Poincaré constant in the space V_0^b . As a byproduct, Theorem 5.5 and the discussions in section 3 yield also a gap in Kolmogorov n -width. It is worth noting that Assumption 5.1 remains largely unexplored, and it is of much interest to further analyze the problem, which we leave to a future work. In the next section, we will present an asymptotic expansion for high-contrast coefficients based on the decomposition (5.7), which verifies the assertion (5.9) and thus yields a low-rank approximation to (1.1) under Assumption 5.1.

6. Asymptotic expansion for high-contrast coefficient case. In this section, we establish the low-rank approximation to (1.1) for high-contrast coefficients, i.e., $\eta \rightarrow \infty$, by means of layer potential techniques and asymptotic expansion. The spectral gap problem has been considered in various settings, e.g., an efficient preconditioner for high-contrast problems, effective conductivity, and multiscale basis functions construction [6, 20, 4, 18, 19]. We shall focus our discussions on the two-dimensional case, and the argument is similar for the three-dimensional case.

6.1. The perfect conductivity problem. The starting point of our analysis is the perfect conductivity problem, whose solution naturally serves as the zeroth order approximation. Specifically, we analyze the solution u_η (where the subscript η emphasizes its dependence on the contrast η) to problem (1.1) with a source term $f \in L^2(D)$ and the coefficient $\kappa := \kappa_\eta$. Upon passing to a subsequence, we have $u_\eta \rightharpoonup u_\infty$ in $H^1(D)$ as $\eta \rightarrow \infty$, where u_∞ is the solution to the following perfect conductivity problem:

$$(6.1) \quad \begin{cases} -\Delta u_\infty = f & \text{in } D_0, \\ u_\infty(x)|_+ = u_\infty(x)|_- & \text{on } \Gamma_i, i = 1, 2, \dots, m, \\ \nabla u_\infty \equiv 0 & \text{in } D_i, i = 1, 2, \dots, m, \\ \int_{\Gamma_i} \frac{\partial u_\infty}{\partial n_i^+} ds(x) = -\int_{D_i} f dx, & i = 1, 2, \dots, m, \\ u_\infty = 0 & \text{on } \partial D. \end{cases}$$

Problem (6.1) can be derived by a variational method along the lines of [2, Appendix]. Further, we can obtain the following a priori estimate:

$$(6.2) \quad \|u_\infty\|_{H^1(D_0)} \leq C_{\text{poin}}(D) \|f\|_{L^2(D)}.$$

Actually, multiplying both sides of the governing equation in (6.1) by u_∞ , integration by parts, and appealing to the interface condition in (6.1) and the fact that u_∞ is piecewise constant on the inclusions $\cup_{i=1}^m \overline{D}_i$ lead directly to

$$\begin{aligned} \|u_\infty\|_{H^1(D_0)}^2 &= -\sum_{i=1}^m \int_{\Gamma_i} \frac{\partial u_\infty}{\partial n_i^+} u_\infty ds(x) + \int_{D_0} u_\infty f dx \\ &= -\sum_{i=1}^m u_\infty \int_{\Gamma_i} \frac{\partial u_\infty}{\partial n_i^+} ds(x) + \int_{D_0} u_\infty f dx \\ &= \sum_{i=1}^m \int_{D_i} u_\infty f dx + \int_{D_0} u_\infty f dx = \int_D u_\infty f dx. \end{aligned}$$

Then Hölder's inequality and the Poincaré inequality yield the desired a priori estimate.

It can be verified that the solution u_∞ to problem (6.1) can be decomposed into

$$u_\infty = w_0 + \sum_{i=1}^m c_i w_i,$$

where c_i are constants that can be uniquely determined through (6.1), the functions $\{w_i\}_{i=1}^m$ are defined in (5.4), and w_0 satisfies

$$\begin{cases} -\Delta w_0 = f & \text{in } D_0, \\ w_0 = 0 & \text{on } \partial D_0. \end{cases}$$

This last problem is commonly known as the perforated problem with a homogeneous Dirichlet boundary condition in the literature. Hölder's inequality and Poincaré inequality imply

$$\|w_0\|_{H^1(D_0)} \leq C_{\text{poin}}(D_0) \|f\|_{L^2(D_0)}$$

with $C_{\text{poin}}(D_0)$ being the Poincaré constant for the domain D_0 .

First, we give a useful orthogonality relation between the difference $u_\eta - u_\infty$ and the space V_m spanned by $\{w_j\}$, defined in (5.4). This result will be used to analyze the leading term approximation below.

LEMMA 6.1. *For the functions w_j , $j = 1, \dots, m$, defined in (5.4), there holds*

$$\int_D \kappa_\eta \nabla(u_\eta - u_\infty) \cdot \nabla w_j dx = 0.$$

Proof. Since w_j is piecewise constant on the domain $D \setminus D_0$, by the divergence theorem, we obtain

$$\begin{aligned} \int_D \kappa_\eta \nabla(u_\eta - u_\infty) \cdot \nabla w_j dx &= \int_{D_0} \kappa_\eta \nabla(u_\eta - u_\infty) \cdot \nabla w_j dx \\ &= - \int_{\Gamma_j} \kappa_\eta \frac{\partial}{\partial n_j^+} (u_\eta - u_\infty) w_j ds(x) - \int_{D_0} \nabla \cdot (\kappa_\eta \nabla(u_\eta - u_\infty)) w_j dx. \end{aligned}$$

By virtue of the governing equations for u_η and u_∞ , the second term on the right-hand side vanishes. For the first term, since $w_j = 1$ on Γ_j and $\kappa_\eta = 1$ in D_0 , we have

$$\int_D \kappa_\eta \nabla(u_\eta - u_\infty) \cdot \nabla w_j dx = - \int_{\Gamma_j} \frac{\partial}{\partial n_j^+} (u_\eta - u_\infty) ds(x).$$

Now the continuity of the flux for u_η on the interface Γ_j and the interface condition for u_∞ in (6.1) imply

$$\begin{aligned} \int_{\Gamma_j} \frac{\partial}{\partial n_j^+} (u_\eta - u_\infty) ds(x) &= \int_{\Gamma_j} \frac{\partial}{\partial n_j^+} u_\eta ds(x) - \int_{\Gamma_j} \frac{\partial}{\partial n_j^+} u_\infty ds(x) \\ &= \int_{\Gamma_j} \kappa_\eta \frac{\partial}{\partial n_j^-} u_\eta ds(x) + \int_{D_j} f dx \\ &= \int_{D_j} (\nabla \cdot (\kappa_\eta \nabla u_\eta) + f) dx = 0, \end{aligned}$$

and this yields the desired result. \square

Let us examine more closely the energy error committed when approximating the solution u_η by the leading term u_∞ . The following energy error follows by a straightforward application of the divergence theorem:

$$(6.3) \quad \begin{aligned} \|u_\eta - u_\infty\|_{H_\kappa^1(D)}^2 &= \langle u_\eta - u_\infty, u_\eta - u_\infty \rangle_D \\ &= \sum_{j=1}^m \int_{\Gamma_j} \left[\kappa_\eta \frac{\partial u_\infty}{\partial n_j} \right] (u_\eta - u_\infty) ds(x) + \sum_{j=1}^m \int_{D_j} f(u_\eta - u_\infty) dx. \end{aligned}$$

This estimate indicates that there are two sources of the energy error: (i) the nonzero source term f on each inclusion D_j and (ii) the mismatch of the interface flux, namely,

$$(6.4) \quad \left[\kappa_\eta \frac{\partial u_\infty}{\partial n_j} \right] = \frac{\partial u_\infty}{\partial n_j^+} \neq 0 \quad \text{on } \Gamma_j \text{ for } j = 1, 2, \dots, m.$$

In order to obtain a good approximation, one has to decrease these two sources of errors, which will be carried out below by means of layer potential techniques and asymptotic expansion.

6.2. Asymptotic expansion. Now we derive a novel asymptotic expansion by carefully analyzing (6.4) using layer potential techniques and asymptotic expansion. This expansion lends itself to a useful low-rank approximation. First, we build auxiliary basis functions to decrease the mismatch on the interfaces. To this end, we denote by $z_j \in L_0^2(\Gamma_j) := \{v \in L^2(\Gamma_j) \text{ with } \int_{\Gamma_j} v ds(x) = 0\}$, the unknown layer potential density for obtaining the auxiliary function in order to decrease the flux mismatch on the interface Γ_j for $j = 1, 2, \dots, m$; cf. (6.4). Let

$$z(x) = \sum_{j=1}^m z_j \delta_{\Gamma_j},$$

and define the operator $\hat{\mathcal{R}} : L^2(D) \rightarrow H_0^1(D)$ by

$$(6.5) \quad \Delta \hat{\mathcal{R}}(z) = z \quad \text{in } D \quad \text{with } \hat{\mathcal{R}}(z) = 0 \quad \text{on } \partial D.$$

Equivalently, $\hat{\mathcal{R}}(z)$ is piecewise harmonic that admits normal jump over the interface Γ_j for $j = 1, 2, \dots, m$, and the function $\hat{\mathcal{R}}(z)$ corresponds to the singular integral over the interfaces Γ_j with densities z_j . Further, we define

$$\mathcal{R}(z, f) := \hat{\mathcal{R}}(z) + \hat{u},$$

where $\hat{u} \in H_0^1(D)$ satisfies

$$(6.6) \quad -\nabla \cdot (\kappa_\eta \nabla \hat{u}) = f \quad \text{in } D_j \quad \text{with } \hat{u} = 0 \quad \text{on } \Gamma_j,$$

and a zero extension on D_0 . Multiplying both sides of (6.6) by \hat{u} , and integrating over the domain D , an application of Hölder's inequality and the Poincaré inequality give

$$\sum_{j=1}^m |\hat{u}|_{H^1(D_j)}^2 = \sum_{j=1}^m \eta_j^{-1} \int_{D_j} f \hat{u} dx \leq \sum_{j=1}^m \eta_j^{-1} C_{\text{poin}}(D_j) \|f\|_{L^2(D_j)} |\hat{u}|_{H^1(D_j)}.$$

Then an application of Young's inequality yields

$$(6.7) \quad |\hat{u}|_{H^1(D)} \leq \max_{j=1,2,\dots,m} \{C_{\text{poin}}(D_j)\eta_j^{-1}\} \|f\|_{L^2(D)}.$$

Therefore, $\|\hat{u}\|_{H^1(D_j)}$ and $\|\frac{\partial \hat{u}}{\partial n_j^-}\|_{H^{-\frac{1}{2}}(\Gamma_j)}$ are both of order $\mathcal{O}(\eta_j^{-1})$, and $\frac{\partial \hat{u}}{\partial n_j^+} = 0$. The solution \hat{u} will be used to correct the force term in the inclusions $\{D_i\}_{i=1}^m$; cf. (6.3).

Next we identify functions $\{z_j\}_{j=1}^m$ such that

$$(6.8) \quad u_\eta = u_\infty + \mathcal{R}(z, f).$$

By the continuity of the flux $\kappa \frac{\partial u_\eta}{\partial n_j}$ across the interface Γ_j and in view of the relation (6.4), this is equivalent to

$$(6.9) \quad \left[\kappa_\eta \frac{\partial}{\partial n_j} \mathcal{R}(z, f) \right] = -\frac{\partial u_\infty}{\partial n_j^+} \quad \text{on } \Gamma_j \text{ for } j = 1, 2, \dots, m.$$

The definition (6.5) indicates that $\hat{\mathcal{R}}$ is harmonic in $D \setminus \bigcup_{j=1}^m \Gamma_j$. Moreover, the next result gives an important characterization of $\hat{\mathcal{R}}(z)$, i.e., $\hat{\mathcal{R}}(z) \in V^h$.

LEMMA 6.2. *For $j = 1, 2, \dots, m$, there holds*

$$\int_{\Gamma_j} \frac{\partial}{\partial n_j^+} \hat{\mathcal{R}}(z) ds(x) = 0.$$

Proof. First, the defining identity (6.5), $\hat{\mathcal{R}}(z)$ is piecewise harmonic, and thus the divergence theorem implies

$$\int_{\Gamma_j} \kappa_\eta \frac{\partial}{\partial n_j^-} \hat{\mathcal{R}}(z) ds(x) = 0.$$

Meanwhile the identity (6.9) and the fact $\frac{\partial \hat{u}}{\partial n_j^+} = 0$ imply

$$(6.10) \quad \left[\kappa_\eta \frac{\partial}{\partial n_j} \hat{\mathcal{R}}(z) \right] = -\frac{\partial u_\infty}{\partial n_j^+} - \left[\kappa_\eta \frac{\partial \hat{u}}{\partial n_j} \right] = -\frac{\partial u_\infty}{\partial n_j^+} + \eta_j \frac{\partial \hat{u}}{\partial n_j^-} \quad \text{on } \Gamma_j.$$

By integrating over Γ_j and applying the divergence theorem, the governing equation (6.6), and the interface condition (6.1), we obtain

$$\int_{\Gamma_j} -\frac{\partial u_\infty}{\partial n_j^+} + \kappa_\eta \frac{\partial \hat{u}}{\partial n_j^-} ds(x) = \int_{D_j} f dx + \int_{D_j} \nabla \cdot (\kappa_\eta \nabla \hat{u}) dx = 0,$$

from which the desired assertion follows. \square

Our main tool to identify the unknown $\{z_j\}_{j=1}^m$ is layer potential techniques. First, we recall a few preliminary results. We denote by $\Phi(x, y) = (2\pi)^{-1} \log |x - y|$ the fundamental solution of the Laplacian in \mathbb{R}^2 . Then the Green's function $G(x, y)$ for the unperturbed domain D is given by

$$G(x, y) = \Phi(x, y) - H(x, y),$$

where $H(x, y)$ represents its regular part satisfying

$$\begin{cases} \Delta_x H(x, y) = 0, & x, y \in D, \\ H(x, y) = (2\pi)^{-1} \log |x - y|, & x \in \partial D, y \in D. \end{cases}$$

Thus, using Green's function $G(x, y)$, the function $\hat{\mathcal{R}}(z)$ admits a (formal) expression

$$(6.11) \quad \hat{\mathcal{R}}(z) = \int_D G(x, y) z(y) dy = \sum_{j=1}^m \left(\int_{\Gamma_j} \Phi(x, y) z_j(y) ds(y) - \int_{\Gamma_j} H(x, y) z_j(y) ds(y) \right).$$

The single layer potential $\mathcal{S}_{D_j} z_j$ of the density function z_j on Γ_j is defined by

$$\mathcal{S}_{D_j} z_j(x) = \int_{\Gamma_j} \Phi(x, y) z_j(y) ds(y),$$

and there hold the well-known jump formula [37],

$$(6.12) \quad \frac{\partial}{\partial n_j^\pm} \mathcal{S}_{D_j} z_j(x) = (\pm \frac{1}{2} + \mathcal{K}_{D_j}^*) z_j(x), \quad x \in \Gamma_j \quad \text{for } j = 1, 2, \dots, m,$$

where $\mathcal{K}_{D_j}^*$ is the $L^2(\Gamma_j)$ -adjoint of the operator \mathcal{K}_{D_j} , defined by

$$\begin{aligned} \mathcal{K}_{D_j} z_j(x) &= \frac{1}{2\pi} \text{p.v.} \int_{\Gamma_j} \frac{(y-x, n_j(y))}{|x-y|^2} z_j(y) ds(y) \\ &:= \frac{1}{2\pi} \lim_{t \rightarrow 0^+} \int_{\Gamma_j \cap |x-y| > t} \frac{(y-x, n_j(y))}{|x-y|^2} z_j(y) ds(y). \end{aligned}$$

Here, p.v. denotes taking the Cauchy principal value. It is well known that if the interface Γ_j is Lipschitz, then the singular integral operator \mathcal{K}_{D_j} is bounded on the space $L^2(\Gamma_j)$ [10]. Further, the identities (6.11) and (6.12) together with the regularity of $H(x, y)$ yield

$$(6.13) \quad \frac{\partial \hat{\mathcal{R}}(z)}{\partial n_j^+} - \frac{\partial \hat{\mathcal{R}}(z)}{\partial n_j^-} = z_j \quad \text{on } \Gamma_j.$$

Next, we choose $\{z_j\}_{j=1}^m$ to satisfy the flux condition (6.9). By the definitions of $\mathcal{R}(z, f)$ and \hat{u} , the flux condition (6.9) is equivalent to (6.10). This relation forms the basis of the asymptotic expansion below. The expression of $\hat{\mathcal{R}}(z)$ in (6.11) and the jump formula (6.12) imply

$$\begin{aligned} \left(\frac{1}{2} z_j - \sum_{i=1}^m \text{p.v.} \int_{\Gamma_i} \frac{\partial G(x, y)}{\partial n_i(x)} z_i(y) ds(y) \right) \eta_j + \frac{1}{2} z_j + \sum_{i=1}^m \text{p.v.} \int_{\Gamma_i} \frac{\partial G(x, y)}{\partial n_i(x)} z_i(y) ds(y) \\ = -\frac{\partial u_\infty}{\partial n_j^+} + \eta_j \frac{\partial \hat{u}}{\partial n_j^-} \quad \text{on } \Gamma_j. \end{aligned}$$

Now we can determine the leading terms of the asymptotic expansion for each $\{z_j\}_{j=1}^m$. Specifically, first, assume that they admit the formal expansion

$$(6.14) \quad z_j(x) = \sum_{\ell=0}^{\infty} z_j^\ell \eta_j^{-\ell}, \quad x \in \Gamma_j,$$

with $z_j^\ell \in L_0^2(\Gamma_j)$ being unknown functions to be determined below.

Further, upon assuming that $\{\eta_j\}_{j=1}^m$ are of comparable magnitude, we let

$$(6.15) \quad z^n(x) = \sum_{j=1}^m \left(\sum_{\ell=0}^n z_j^\ell \eta_j^{-\ell} \right) \delta_{\Gamma_j}$$

be the n th-order approximation to z . Then the n th-order approximation u^n to u_η is defined by

$$(6.16) \quad u^n = u_\infty + \hat{u} + \hat{\mathcal{R}}(z^n).$$

Upon substituting (6.14) into (6.10) and collecting terms according to the order in η_j , by the trace formula and Lemma 6.2, we obtain the following hierarchies:

(i) the $\mathcal{O}(\eta)$ term,

$$(6.17) \quad \frac{\partial}{\partial n_j^-} \hat{\mathcal{R}}(z^0) = 0 \quad \text{and} \quad \int_{\Gamma_j} \frac{\partial}{\partial n_j^+} \hat{\mathcal{R}}(z^0) ds(x) = 0;$$

(ii) the $\mathcal{O}(1)$ term,

$$\begin{cases} -\frac{\partial}{\partial n_j^-} \hat{\mathcal{R}}(z^1 - z^0) \eta_j + \frac{\partial}{\partial n_j^+} \hat{\mathcal{R}}(z^0) = -\frac{\partial u_\infty}{\partial n_j^+} + \eta_j \frac{\partial \hat{u}}{\partial n_j^-}, \\ \int_{\Gamma_j} \frac{\partial}{\partial n_j^+} \hat{\mathcal{R}}(z^1) ds(x) = 0; \end{cases}$$

(iii) the high-order terms, for $\ell = 1, 2, \dots$, the $\mathcal{O}(\eta^{-\ell})$ term,

$$(6.18) \quad \begin{cases} -\frac{\partial}{\partial n_j^-} \hat{\mathcal{R}}(z^{\ell+1} - z^\ell) \eta_j + \frac{\partial}{\partial n_j^+} \hat{\mathcal{R}}(z^\ell - z^{\ell-1}) = 0, \\ \int_{\Gamma_j} \frac{\partial}{\partial n_j^+} \hat{\mathcal{R}}(z^{\ell+1}) ds(x) = 0. \end{cases}$$

Next we discuss these terms one by one. First, for the $\mathcal{O}(\eta)$ term, the homogeneous Neumann boundary condition in (6.17) and the fact that $\hat{\mathcal{R}}(z_0)$ is harmonic over D_j imply that $\hat{\mathcal{R}}(z_0)$ is constant on Γ_j , and thus $\hat{\mathcal{R}}(z^0) \in V_m \cap V^h$; cf. Lemma 6.2. Then Theorem 5.4 yields

$$z_j^0 = 0, \quad j = 1, 2, \dots, m.$$

Next, we solve for the second term z^1 , which satisfies

$$(6.19) \quad \frac{\partial}{\partial n_j^-} \hat{\mathcal{R}}(z^1) = \eta_j^{-1} \left(\frac{\partial u_\infty}{\partial n_j^+} - \eta_j \frac{\partial \hat{u}}{\partial n_j^-} \right), \quad j = 1, 2, \dots, m.$$

The identity (6.1) together with (6.6) gives

$$\int_{\Gamma_j} \left(\frac{\partial u_\infty}{\partial n_j^+} - \eta_j \frac{\partial \hat{u}}{\partial n_j^-} \right) ds(x) = - \int_{D_j} f dx - \left(- \int_{D_j} f dx \right) = 0, \quad j = 1, \dots, m.$$

Therefore, the second term $\hat{\mathcal{R}}(z^1)$ inside each inclusion D_j can be obtained by solving

$$\begin{cases} -\nabla \cdot (\kappa_\eta \nabla \hat{\mathcal{R}}(z^1)) = 0 & \text{in } D_j, \\ \frac{\partial}{\partial n_j^-} \hat{\mathcal{R}}(z^1) = \eta_j^{-1} \left(\frac{\partial u_\infty}{\partial n_j^+} - \eta_j \frac{\partial \hat{u}}{\partial n_j^-} \right) & \text{on } \Gamma_j. \end{cases}$$

Then the value of $\hat{\mathcal{R}}(z^1)$ in D_0 can be attained by the Harmonic extension.

The higher-order terms $\hat{\mathcal{R}}(z^\ell)$ for $\ell = 2, 3, \dots$ inside each inclusion D_j are determined by their Neumann data directly, which in turn is related to the Neumann data of the lower-order terms in D_0 by (6.18). The Dirichlet data of the latter is available by the continuity of $\hat{\mathcal{R}}(z^\ell)$ along the interface Γ_j for $j = 1, 2, \dots, m$. Thus, we employ the DtN map and NtD map. We denote by $\Lambda_j^N : H^{-\frac{1}{2}}(\Gamma_j) \rightarrow H^{\frac{1}{2}}(\Gamma_j)$ the NtD map on D_j and by $\Lambda^D : H^{\frac{1}{2}}(\partial D_0) \rightarrow H^{-\frac{1}{2}}(\partial D_0)$ the DtN map on D_0 . Then the Neumann data of lower orders in D_0 can be expressed as

$$\begin{aligned} & \left[\frac{\partial}{\partial n_1^+} \hat{\mathcal{R}}(z^\ell - z^{\ell-1}), \frac{\partial}{\partial n_2^+} \hat{\mathcal{R}}(z^\ell - z^{\ell-1}), \dots, \frac{\partial}{\partial n_m^+} \hat{\mathcal{R}}(z^\ell - z^{\ell-1}) \right] \\ &= \Lambda^D(\hat{\mathcal{R}}(z^\ell - z^{\ell-1})), \quad \ell = 1, 2, \dots \end{aligned}$$

Together with (6.13), this yields $z_j^1 \in L_0^2(\Gamma_j)$. The boundedness of the operators Λ_j^N and Λ^D implies

$$\begin{aligned} (6.20) \quad & \left(\sum_{j=1}^m \left\| \frac{\partial}{\partial n_j^+} \hat{\mathcal{R}}(z^\ell - z^{\ell-1}) \right\|_{H^{-\frac{1}{2}}(\Gamma_j)}^2 \right)^{\frac{1}{2}} \\ & \leq \|\Lambda^D\| \max_{j=1, \dots, m} \{\|\Lambda_j^N\|\} \left(\sum_{j=1}^m \left\| \frac{\partial}{\partial n_j^-} \hat{\mathcal{R}}(z^\ell - z^{\ell-1}) \right\|_{H^{-\frac{1}{2}}(\Gamma_j)}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Then we obtain the higher-order terms $\hat{\mathcal{R}}(z^{\ell+1})$ by solving Neumann problems in D_j :

$$-\Delta \hat{\mathcal{R}}(z^{\ell+1}) = 0 \quad \text{in } D_j,$$

together with the corresponding boundary condition

$$\begin{aligned} (6.21) \quad & \frac{\partial}{\partial n_j^-} \hat{\mathcal{R}}(z^{\ell+1}) = \frac{\partial}{\partial n_j^-} \hat{\mathcal{R}}(z^\ell) + \eta_j^{-1} \frac{\partial}{\partial n_j^+} \hat{\mathcal{R}}(z^\ell - z^{\ell-1}) \quad \text{on } \Gamma_j \\ & \text{satisfying } \int_{\Gamma_j} \frac{\partial}{\partial n_j^-} \hat{\mathcal{R}}(z_j^{\ell+1}) ds(x) = 0, \end{aligned}$$

which is a consequence of the higher-order terms in (6.18), (6.13) and the fact that z_j^ℓ and $z_j^{\ell-1}$ belong to $L_0^2(\Gamma_j)$. Clearly, this is a well-posed problem. Next, we bound the energy error $\|u_\eta - u^n\|_{H_\kappa^1(D)}$. To this end, we first derive the expression of the flux jump of u^n .

LEMMA 6.3. *Let u^n be the n th-order approximation to u_η defined in (6.16) for $n \in \mathbb{N}_+$. Then there holds*

$$\left[\kappa_\eta \frac{\partial u^n}{\partial n_j} \right] = \frac{\partial}{\partial n_j^+} \hat{\mathcal{R}}(z^n - z^{n-1}) \quad \text{on } \Gamma_j, \quad j = 1, \dots, m.$$

Proof. By the definition of u^n in (6.16) and noting $\frac{\partial u_\infty}{\partial n_j} = 0$, we have

$$\eta_j \frac{\partial u^n}{\partial n_j^-} = \eta_j \frac{\partial u_\infty}{\partial n_j^-} + \eta_j \frac{\partial}{\partial n_j^-} \hat{\mathcal{R}}(z^n) + \eta_j \frac{\partial \hat{u}}{\partial n_j^-} = \eta_j \frac{\partial}{\partial n_j^-} \hat{\mathcal{R}}(z^n) + \eta_j \frac{\partial \hat{u}}{\partial n_j^-},$$

Then by writing $\frac{\partial}{\partial n_j^-} \hat{\mathcal{R}}(z^n)$ as a telescopic sum and using (6.19) and (6.21), we obtain

$$\begin{aligned}\eta_j \frac{\partial u^n}{\partial n_j^-} &= \eta_j \frac{\partial \hat{\mathcal{R}}(z^1)}{\partial n_j^-} + \eta_j \sum_{\ell=2}^n \frac{\partial}{\partial n_j^-} \hat{\mathcal{R}}(z^\ell - z^{\ell-1}) + \eta_j \frac{\partial \hat{u}}{\partial n_j^-} \\ &= \frac{\partial u_\infty}{\partial n_j^+} + \sum_{\ell=1}^{n-1} \frac{\partial}{\partial n_j^+} \hat{\mathcal{R}}(z^\ell - z^{\ell-1}).\end{aligned}$$

Likewise, by the definition of u^n , and noting $\frac{\partial \hat{u}}{\partial n_j^+} = 0$ and $\frac{\partial}{\partial n_j^+} \hat{\mathcal{R}}(z^0) = 0$ (since $\hat{\mathcal{R}}(z^0) = 0$), a direct calculation leads to

$$\frac{\partial u^n}{\partial n_j^+} = \frac{\partial u_\infty}{\partial n_j^+} + \frac{\partial \hat{u}}{\partial n_j^+} + \frac{\partial}{\partial n_j^+} \hat{\mathcal{R}}(z^n) = \frac{\partial u_\infty}{\partial n_j^+} + \sum_{\ell=1}^n \frac{\partial}{\partial n_j^+} \hat{\mathcal{R}}(z^\ell - z^{\ell-1}).$$

Now the desired result follows by subtraction the preceding two identities. \square

A similar argument as for (6.3) together with Lemma 6.3 yields

$$\begin{aligned}\|u_\eta - u^n\|_{H_\kappa^1(D)}^2 &= \langle u_\eta - u^n, u_\eta - u^n \rangle_D = \sum_{j=1}^m \int_{\Gamma_j} \left[\kappa_\eta \frac{\partial u^n}{\partial n_j} \right] (u_\eta - u^n) ds(x) \\ (6.22) \quad &= \sum_{j=1}^m \int_{\Gamma_j} \frac{\partial}{\partial n_j^+} \hat{\mathcal{R}}(z^n - z^{n-1}) (u_\eta - u^n) ds(x).\end{aligned}$$

The next lemma estimates the first term in the integral of the last equation in (6.22).

LEMMA 6.4. *Let z^n be defined in (6.15) with $n \in \mathbb{N}_+$. There holds*

$$\begin{aligned}(6.23) \quad &\sum_{j=1}^m \left\| \frac{\partial}{\partial n_j^+} \hat{\mathcal{R}}(z^n - z^{n-1}) \right\|_{H^{-\frac{1}{2}}(\Gamma_j)}^2 \\ &\lesssim \eta_{\min}^{-2n} (C_{\text{poin}}(D))^2 + \max\{C_{\text{poin}}(D_j)^2\} \|f\|_{L^2(D)}^2.\end{aligned}$$

Proof. We prove the result by mathematical induction. First we consider the case $n = 1$. In view of $\hat{\mathcal{R}}(z^0) = 0$, by (6.20) and the flux condition (6.19), we have

$$\sum_{j=1}^m \left\| \frac{\partial \hat{\mathcal{R}}(z^1)}{\partial n_j^+} \right\|_{H^{-\frac{1}{2}}(\Gamma_j)}^2 \lesssim \sum_{j=1}^m \left\| \frac{\partial \hat{\mathcal{R}}(z^1)}{\partial n_j^-} \right\|_{H^{-\frac{1}{2}}(\Gamma_j)}^2 = \sum_{j=1}^m \eta_j^{-2} \left\| \frac{\partial u_\infty}{\partial n_j^+} - \eta_j \frac{\partial \hat{u}}{\partial n_j^-} \right\|_{H^{-\frac{1}{2}}(\Gamma_j)}^2.$$

By the trace theorem and the a priori estimate (6.2),

$$\sum_{j=1}^m \left\| \frac{\partial u_\infty}{\partial n_j^+} \right\|_{H^{-\frac{1}{2}}(\Gamma_j)}^2 \lesssim \int_{D_0} |\nabla u_\infty|^2 dx \leq C_{\text{poin}}(D)^2 \|f\|_{L^2(D)}^2.$$

Likewise, by the trace theorem and the a priori estimate (6.7), we deduce

$$\sum_{j=1}^m \left\| \eta_j \frac{\partial \hat{u}}{\partial n_j^-} \right\|_{H^{-\frac{1}{2}}(\Gamma_j)}^2 \lesssim \sum_{j=1}^m \eta_j^2 |\hat{u}|_{H^1(D_j)}^2 \leq \max\{C_{\text{poin}}(D_j)^2\} \|f\|_{L^2(D)}^2.$$

Combining the last three estimates yields

$$\sum_{j=1}^m \left\| \frac{\partial \hat{\mathcal{R}}(z^1)}{\partial n_j^+} \right\|_{H^{-\frac{1}{2}}(\Gamma_j)}^2 \lesssim \eta_{\min}^{-2} (C_{\text{poin}}(D)^2 + \max\{C_{\text{poin}}(D_j)^2\}) \|f\|_{L^2(D)}^2.$$

This verifies (6.23) for $n = 1$. Now assume that it holds for some $n = \ell > 1$, and we show that (6.23) holds for $n = \ell + 1$. Appealing to (6.20) and (6.21) yields

$$\begin{aligned} \sum_{j=1}^m \left\| \frac{\partial}{\partial n_j^+} \hat{\mathcal{R}}(z^{\ell+1} - z^\ell) \right\|_{H^{-\frac{1}{2}}(\Gamma_j)}^2 &\lesssim \sum_{j=1}^m \left\| \frac{\partial}{\partial n_j^-} \hat{\mathcal{R}}(z^{\ell+1} - z^\ell) \right\|_{H^{-\frac{1}{2}}(\Gamma_j)}^2 \\ &= \sum_{j=1}^m \eta_j^{-2} \left\| \frac{\partial}{\partial n_j^+} \hat{\mathcal{R}}(z^\ell - z^{\ell-1}) \right\|_{H^{-\frac{1}{2}}(\Gamma_j)}^2 \\ &\lesssim \eta_{\min}^{-2(\ell+1)} (C_{\text{poin}}(D)^2 + \max\{C_{\text{poin}}(D_j)^2\}) \|f\|_{L^2(D)}^2, \end{aligned}$$

where the last line follows from the induction hypothesis, completing the proof. \square

By the elliptic regularity theory and Lemma 6.4, the following assertion holds.

PROPOSITION 6.5. *Let the n th-order approximation z^n be defined in (6.15) for $n \in \mathbb{N}_+$. Then there holds*

$$(6.24) \quad \left\| \hat{\mathcal{R}}(z^n - z^{n-1}) \right\|_{H_\kappa^1(D)} \lesssim \eta_{\min}^{-n+\frac{1}{2}} (C_{\text{poin}}(D) + \max\{C_{\text{poin}}(D_j)\}) \|f\|_{L^2(D)}.$$

Proof. By the elliptic regularity in the domain D_0 and each inclusion D_i and Lemma 6.4, we deduce

$$\begin{aligned} \left\| \hat{\mathcal{R}}(z^n - z^{n-1}) \right\|_{H_\kappa^1(D)}^2 &= \left| \hat{\mathcal{R}}(z^n - z^{n-1}) \right|_{H^1(D_0)}^2 + \sum_{i=1}^m \eta_i \left| \hat{\mathcal{R}}(z^n - z^{n-1}) \right|_{H^1(D_i)}^2 \\ &\lesssim \sum_{i=1}^m \left\| \frac{\partial}{\partial n_i^+} \hat{\mathcal{R}}(z^n - z^{n-1}) \right\|_{H^{-\frac{1}{2}}(\Gamma_i)}^2 + \sum_{i=1}^m \eta_i \left\| \frac{\partial}{\partial n_i^-} \hat{\mathcal{R}}(z^n - z^{n-1}) \right\|_{H^{-\frac{1}{2}}(\Gamma_i)}^2 \\ &\lesssim \eta_{\min}^{-2n+1} (C_{\text{poin}}(D)^2 + \max\{C_{\text{poin}}(D_j)^2\}) \|f\|_{L^2(D)}^2. \end{aligned}$$

The assertion follows by taking the square root of both sides. \square

Finally, we are ready to state an energy error estimate by combining (6.22) with Lemma 6.4 and show its proof.

THEOREM 6.6. *Let u^n be the n th-order approximation to u_η defined in (6.16). There holds*

$$\|u_\eta - u^n\|_{H_\kappa^1(D)} \lesssim \eta_{\min}^{-n} (C_{\text{poin}}(D) + \max\{C_{\text{poin}}(D_j)\}) \|f\|_{L^2(D)}$$

with $\eta_{\min} := \min\{\eta_j, j = 1, 2, \dots, m\}$.

Proof. By (6.22) and the trace theorem

$$\begin{aligned} \|u_\eta - u^n\|_{H_\kappa^1(D)}^2 &= \sum_{j=1}^m \int_{\Gamma_j} \frac{\partial}{\partial n_j^+} \hat{\mathcal{R}}(z^n - z^{n-1}) (u_\eta - u^n) ds(x) \\ &\lesssim \sum_{j=1}^m \left\| \frac{\partial}{\partial n_j^+} \hat{\mathcal{R}}(z^n - z^{n-1}) \right\|_{H^{-\frac{1}{2}}(\Gamma_j)} \|u_\eta - u^n\|_{H^{\frac{1}{2}}(\Gamma_j)}. \end{aligned}$$

The assertion follows from Lemma 6.4, Hölder's inequality, and that $\eta_{\min} \geq 1$. \square

The asymptotic expansion for high-contrast problems when $\eta \rightarrow \infty$ was studied earlier [8, 7]. However, our result contains a much better zeroth-order approximation, i.e., the solution u_∞ to the perfect conductivity problem (6.1), which is the weak limit of u_η in $H^1(D)$ as $\eta \rightarrow \infty$, and thus also a much sharper error estimate.

PROPOSITION 6.7. *Let $\eta \rightarrow \infty$. There holds*

$$\|u_\eta - u_\infty\|_{H^1_\kappa(D)} \lesssim \eta_{\min}^{-\frac{1}{2}} (C_{\text{poin}}(D) + \max\{C_{\text{poin}}(D_j)\}) \|f\|_{L^2(D)}.$$

Proof. This result follows from Theorem 6.6, Proposition 6.5 for $n = 1$, the a priori estimate (6.7) and the triangle inequality. \square

Last, we examine the connection between the n th approximant u^n in (6.16) and the orthogonal decomposition (5.7) more closely. Note that $u_\infty \in V_m \oplus V_0^b$, $\hat{u} \in V^b$ and $\hat{\mathcal{R}}(z^n) \in V^h$. The zeroth-order approximant u_∞ is related to the force term f via the component w_0 , the term \hat{u} also depends on f (cf. (6.6)), and the dependence of $\hat{\mathcal{R}}(z^n)$ on f is due to the normal flux (6.18). In order to obtain a low-rank approximation to u_η that is independent of the force term f (cf. (1.4)), we apply Assumption 5.1. Propositions 3.2 and 6.7 and Theorem 5.3 yield directly Proposition 6.8.

PROPOSITION 6.8. *Let $d = 2$, and let Assumption 5.1 be valid. Assume that $\eta \rightarrow \infty$ and $\delta_j \gg \epsilon_i$ for $j = 1, 2, \dots, m$. There holds*

$$d_i(\mathcal{S}(W); V) \begin{cases} \geq \sqrt{\frac{|D_{i+1}|}{\pi}} \text{ for } i \leq m-1; \\ \lesssim \eta_{\min}^{-\frac{1}{2}} (C_{\text{poin}}(D) + \max\{C_{\text{poin}}(D_j)\}) \text{ for } i = m. \end{cases}$$

Remark 6.1. First, Proposition 6.8 implies the assertion (5.9). Indeed, an immediate corollary of Proposition 3.2 implies

$$(6.25) \quad \lambda_{m+1} \lesssim \eta_{\min}^{-1}.$$

Next, we show (5.9) by contradiction. Assume that (5.9) does not hold; then there exists $v \in V^h$ such that $R(v) \gg \eta_{\min}^{-1}$. Let $X_{m+1} := V_m \oplus Y$ with $Y \subset V^h$ being the one-dimensional linear space spanned by v . Then Theorem 5.3 and the orthogonality of V_m and Y imply $\min_{v \in X_{m+1}} R(V) \gg \eta_{\min}^{-1}$, which contradicts (6.25) in view of (5.3). Hence, the assertion (5.9) is proved.

Further, it indicates that there is a spectral gap in the high-contrast limit, i.e., as $\eta \rightarrow \infty$, if Assumption 5.1 holds. Moreover, there are precisely m dominant eigenmodes, where m is the number of inclusions. Such a gap implies the existence of an effective low-rank approximation and can and should be effectively employed in the numerical treatment of high-contrast problems.

7. Conclusion. In this work, we have investigated the low-rank approximation properties to heterogeneous elliptic problems and provided their optimal approximation rate via the concept of Kolmogorov n -width, which is essentially related to the eigenvalue decay rate of the solution map. To illustrate the important role the structure of the coefficient plays in the low-rank property of the solution, we provided one sufficient condition for low-rank approximation, which directly motivates the use of harmonic functions. In order to derive the eigenvalue decay rate, we discussed realistic assumptions on the permeability field κ , e.g., the values, the locations of the inclusions, and the pairwise distances, which would hugely influence the eigenvalues. Further, we have provided a new eigenvalue estimate for elliptic operators with

a high-contrast coefficient and derived a new asymptotic expansion with respect to the high-contrast coefficient, which are of independent interest. These results show the existence of a low-rank structure of the solution manifold for certain heterogeneous problems and thereby provide the theoretical justifications of multiscale model reduction techniques.

This work represents a first step toward the complete theoretical understanding of multiscale model reduction algorithms. There are a few lines for future research, e.g., general L^∞ coefficient, low-conductivity inclusions, and optimal approximation rate. For example, the asymptotic expansion is formally applicable to the case of low-conductivity inclusions; however, the existence of a limit problem remains to be shown. Numerically, it is of immense interest to turn the theoretical results into constructive multiscale model reduction algorithms (with provable optimal computational complexity). One promising step to leverage the analytical results in section 6 is to derive refined characterizations of the solution space of the perforated problem.

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